Unconditionally Stable Current Density Convolution Crank–Nicolson Finite-Difference Time-Domain Implementation for Anisotropic Magnetized Plasma

Jianxiong Li
School of Electronics and Information Engineering, Tianjin Polytechnic University, Tianjin, China
Tianjin Key Laboratory of Optoelectronic Detection Technology and Systems, Tianjin, China
Email: lijianxiong@tjpu.edu.cn

Yongjia Zhuang
School of Electronics and Information Engineering, Tianjin Polytechnic University, Tianjin, China
Tianjin Key Laboratory of Optoelectronic Detection Technology and Systems, Tianjin, China

***Abstract*—An effective unconditionally stable implementation of the current density convolution Crank–Nicolson finite-difference time-domain (JEC-CN-FDTD) method for anisotropic magnetized plasma is proposed. The JEC-CN-FDTD method for isotropic dispersive media greatly improves efficiency and retains its accuracy. This paper extends this approach to anisotropic magnetized plasma. This method not only solves the problem that incorporates both anisotropy and frequency dispersion at the same time, but also eliminates the Courant–Friedrich–Levy (CFL) stability constraint. A numerical example has been carried out to validate the proposed formulations in one dimension of electromagnetic wave through anisotropic magnetized plasma slab. The results prove that the proposed formulations significantly save time and perform stably with acceptable accuracy.**

***Index Terms*—Crank–Nicolson difference scheme (CN), current density convolution (JEC), finite-difference time-domain (FDTD), magnetized plasma**

I. Introduction

The finite-different time-domain (FDTD) method is one of the most popular methods for the solution of problems in analysis and design of electromagnetic propagation, microwave structures, and many other engineering applications. Over the past years, the FDTD method has been widely used to simulate dispersive media including the anisotropic magnetized plasma media [1]. However, the conventional FDTD method is conditionally stable, namely, the time step is restricted by the Courant-Friedrichs-Lewy (CFL) stability condition [2]. When a small spatial step is required for fine geometrical details, the time step has to be small so that the simulation time is too long to be accepted. To overcome the CFL stability limit of the FDTD method, some unconditionally stable FDTD algorithms including the alternating-direction-implicit (ADI) FDTD [3], Crank–Nicolson (CN) FDTD [4], split-step FDTD [5] and locally one-dimensional (LOD) FDTD [6] have been introduced.

Recently, the current density convolution CN–FDTD (JEC-CN-FDTD) method has been successfully applied to one-dimensional (1-D) unmagnetized plasma medium based on incomplete Cholesky conjugate gradient (ICCG) method [7]. In this paper, the JEC-CN-FDTD method is extended to simulation 1-D anisotropic magnetized plasma without using ICCG method. A main advantage over the method in [7] is that the CN equations are solved, as naturally expected, by two tridiagonal algorithms, requiring neither preconditioners, nor iterative solvers, to deal with the implicit equations. And the proposed difference iterative formulations of anisotropic magnetized plasma eliminate the CFL stability constraint based on the CN method. Therefore, the selection of time step is not limited by the CFL stability condition. Under the sufficient accuracy of the calculation, the number of the simulation steps can be reduced with increasing the time step, which can lead to the substantially decline of the computing time. The high accuracy and efficiency of the proposed JEC-CN-FDTD algorithm are confirmed by computing reflection and transmission of a magnetized plasma slab. Comparing the JEC-CN-FDTD method with the conventional JEC-FDTD method [8] and analytical solutions, the results show that the proposed formulations can reduce most of computing time, perform stably and facilitate computer programming, but retain its accuracy.

II. Formulations

In anisotropic magnetized plasma medium with collisions, supposing that the external static magnetic field is parallel to the z axis, the Maxwell's component equations are written as:

$$\varepsilon_0 \frac{\partial E_x}{\partial t} + J_x = -\frac{\partial H_y}{\partial z} \tag{1}$$

$$\varepsilon_0 \frac{\partial E_y}{\partial t} + J_y = \frac{\partial H_x}{\partial z} \tag{2}$$

$$\mu_0 \frac{\partial H_x}{\partial t} = \frac{\partial E_y}{\partial z} \tag{3}$$

$$\mu_0 \frac{\partial H_y}{\partial t} = -\frac{\partial E_x}{\partial z} \tag{4}$$

where $E_\eta(\eta = x, y)$ is the electric field intensity in the η direction, H_η is the magnetic field intensity, J_η is the polarization current density, ε_0 and μ_0 are the permittivity and permeability coefficient of free space, respectively. The constitutive relationships of polarization current density J_x and J_y are given by

$$\frac{\partial J_x}{\partial t} + vJ_x = \varepsilon_0 \omega_p^2 E_x - \omega_b J_y \tag{5}$$

$$\frac{\partial J_y}{\partial t} + vJ_y = \varepsilon_0 \omega_p^2 E_y + \omega_b J_x \tag{6}$$

where v is the electron collision frequency, ω_p the plasma frequency, and ω_b the electron gyrofrequency. The plasma medium parameters are constant that do not vary with time and anisotropy. From (5) and (6), it is obvious that J_x and J_y are coupled. Therefore, the update equations for two components of polarization current density need to be solved simultaneously. According to the conventional JEC-FDTD method presented in [8], (7) and (8) can be obtained as

$$J_x^{n+1/2}(k) = b_1 J_x^{n-1/2}(k) + b_2 E_x^n(k) - b_3 J_y^{n-1/2}(k) - b_4 E_y^n(k) \tag{7}$$

$$J_y^{n+1/2}(k) = b_1 J_y^{n-1/2}(k) + b_2 E_y^n(k) + b_3 J_x^{n-1/2}(k) + b_4 E_x^n(k) \tag{8}$$

where

$b_1 = \dfrac{(4-\omega_b^2 \Delta t^2) e^{-v\Delta t}}{b_5}$, $b_2 = \dfrac{4\varepsilon_0 \omega_p^2 \Delta t e^{-v\Delta t/2}}{b_5}$,

$b_3 = \dfrac{b_6}{b_5}(1+e^{-v\Delta t})$, $b_4 = \dfrac{b_6}{b_5} \cdot \varepsilon_0 \omega_p^2 \Delta t e^{-v\Delta t/2}$,

$b_5 = 4 + \omega_b^2 \Delta t^2 e^{-v\Delta t}$, $b_6 = 2\omega_b \Delta t e^{-v\Delta t/2}$,

where Δt is the time step, k is the spatial location index.

The CN scheme is used in spatial partial differential (1)-(4). For example, (1) and (4) can be formulated as follows:

$$E_x^{n+1}(k) = E_x^n(k) - \frac{\Delta t}{2\varepsilon_0 \Delta z}[H_y^{n+1}(k+\tfrac{1}{2}) - H_y^{n+1}(k-\tfrac{1}{2}) + H_y^n(k+\tfrac{1}{2}) - H_y^n(k-\tfrac{1}{2})] - \frac{\Delta t}{\varepsilon_0} J_x^{n+1/2}(k) \tag{9}$$

$$H_y^{n+1}(k+\tfrac{1}{2}) = H_y^n(k+\tfrac{1}{2}) - \frac{\Delta t}{2\mu_0 \Delta z}[E_x^{n+1}(k+1) - E_x^{n+1}(k) + E_x^n(k+1) - E_x^n(k)] \tag{10}$$

where Δz is the space step in the z direction.

It is noted that the discrete electric and magnetic field components are coupled implicitly, which leads to a huge sparse matrix to be solved. To decouple them, substituting (10) into (9) and after some manipulations, an implicit update for E_x^{n+1} is obtained from

$$-a_z E_x^{n+1}(k-1) + (1+2a_z) E_x^{n+1}(k) - a_z E_x^{n+1}(k+1) = a_z E_x^n(k-1) + (1-2a_z) E_x^n(k) + a_z E_x^n(k+1) - \frac{\Delta t}{\varepsilon_0 \Delta z}\left(H_y^n(k+\tfrac{1}{2}) - H_y^n(k-\tfrac{1}{2})\right) - \frac{\Delta t}{\varepsilon_0} J_x^{n+1/2}(k) \tag{11}$$

where

$$a_z = \frac{\Delta t^2}{4\varepsilon_0 \mu_0 \Delta z^2}$$

By the same procedure, E_y^{n+1} and H_x^{n+1} can be updated as follows:

$$-a_z E_y^{n+1}(k-1) + (1+2a_z) E_y^{n+1}(k) - a_z E_y^{n+1}(k+1) = a_z E_y^n(k-1) + (1-2a_z) E_y^n(k) + a_z E_y^n(k+1) + \frac{\Delta t}{\varepsilon_0 \Delta z}\left(H_x^n(k+\tfrac{1}{2}) - H_x^n(k-\tfrac{1}{2})\right) - \frac{\Delta t}{\varepsilon_0} J_y^{n+1/2}(k) \tag{12}$$

$$H_x^{n+1}(k+\tfrac{1}{2}) = H_x^n(k+\tfrac{1}{2}) + \frac{\Delta t}{2\mu_0 \Delta z}[E_y^{n+1}(k+1) - E_y^{n+1}(k) + E_y^n(k+1) - E_y^n(k)] \tag{13}$$

It is clear that the left-hand sides of (11) and (12) form two tri-diagonal matrix which can be solved for E_x^{n+1} and E_y^{n+1} easily. And, H_x^{n+1}, H_y^{n+1}, $J_x^{n+1/2}$ and $J_y^{n+1/2}$ are updated explicitly. It is remarkable that, because of using the JEC method, polarization current density J_x and J_y are calculated at half integer time step (n+1/2), which makes entire update formulations and the computer programming simple.

III. NUMERICAL STUDY

A numerical example that the electromagnetic wave propagates through a magnetized collision plasma slab is used to validate the proposed JEC-CN-FDTD formulations. The parameters of the plasma are taken as $\omega_p = 2\pi \times 28.7 \times 10^9$ rad/s, $\omega_b = 1.0 \times 10^{11}$ rad/s, and $v = 20 \times 10^9$ rad/s. The thickness of the plasma slab is 15mm, and the incident wave used in the simulation is a differential Gaussian pulse with the peak frequency of 50 GHz.

To improve the accuracy and decrease the numerical dispersive error, CPW =100 is chosen, where CPW denotes the number of the cell per wavelength. In the numerical example, the $\mathrm{CFLN} \le 6$ is chosen, which is defined as $\mathrm{CFLN} = \Delta t / \Delta t_{CFL}^{FDTD}$, where Δt_{CFL}^{FDTD} is the maximum stability limit of the conventional FDTD algorithm, in this test, $\Delta t_{CFL}^{FDTD} = 0.1$ ps. It is noted that time sampling precision is high enough because of $\mathrm{CPW/CFLN>12}$. The computational domain has 2000 cells in the z direction. The space step is $\Delta z = 30\ \mu\mathrm{m}$, the plasma slab occupies 500 cells in the middle of the FDTD domain, and the rest is free space. Eight-cell perfectly matched layer (PML) is used at two terminations of the space to eliminate unwanted reflections [9]. In this test, the simulation is carried out for the first 1.6384 ns.

As shown in Figs. 1-4, the magnitudes of reflection coefficients and transmission coefficients for RCP and LCP waves are computed using the proposed JEC-CN-FDTD method with different CFLNs. At the same time, these results obtained from conventional JEC-FDTD method [8] and analytical solutions are given.

Fig. 1 and Fig. 2 show that the reflection coefficients from the proposed JEC-CN-FDTD formulations with different CFLNs keep a high accuracy, which is close to the analytical solutions. When CFLN = 6, there is a tiny deviation at high frequency, but this deviation is acceptable.

Fig. 3 and Fig. 4 show that the transmission coefficients from the proposed JEC-CN-FDTD algorithm with different CFLNs follow the analytical solutions closely at total interesting frequency range.

Fig. 1-4 confirm that the proposed JEC-CN-FDTD formulations for anisotropic magnetized plasma are valid and unconditionally stable, and its simulation results with different CFLNs keep in good agreement with analytical solutions.

As shown in Table I, it is obvious that the JEC-CN-FDTD method occupies larger memory than the conventional JEC-FDTD method in [8], but the JEC-CN-FDTD method saves more and more time with the increasing of CFLN value when CFLN $\geq$ 2 at the acceptable cost of the memory. Especially, this method saves more than 76% time when CFLN=6. In this test, a PC with Inter(R) core(TM) i7 CPU @ 2.6GHz and 8GB(DDR4 2133MHz) memory is used.

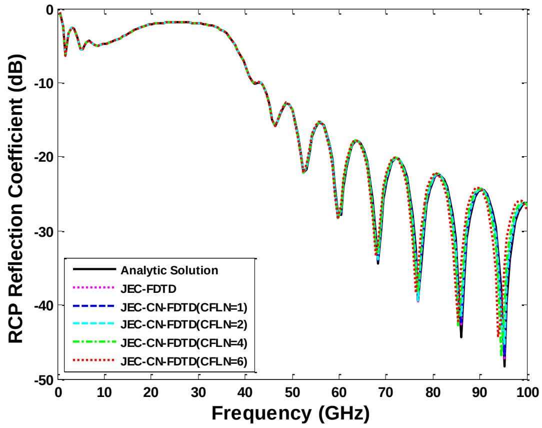

Figure 1. RCP reflection coefficients versus frequency obtained from the proposed JEC-CN-FDTD formulations with different CFLNs, the conventional JEC-FDTD and analytical solution for anisotropic magnetized plasma.

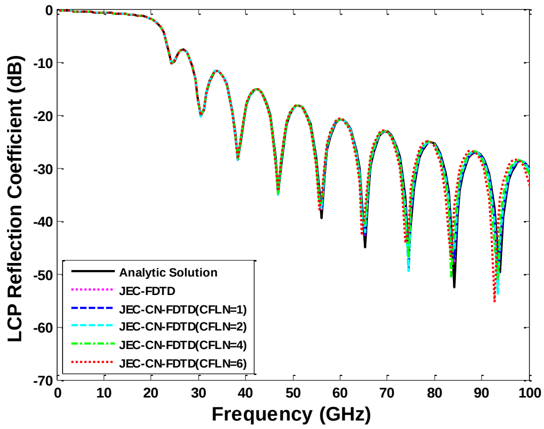

Figure 2. LCP reflection coefficients versus frequency obtained from the proposed JEC-CN-FDTD formulations with different CFLNs, the conventional JEC-FDTD and analytical solution for anisotropic magnetized plasma.

IV. CONCLUSION

In this paper, the JEC-CN-FDTD algorithm for anisotropic magnetized plasma is proposed. The numerical example shows that the proposed algorithm maintains unconditional stability and the time step is not limited by the CFL stability condition. In addition, the proposed algorithm can keep a high accuracy and consume very little time.

TABLE I.
TIME AND MEMORY USED BY CONVENTIONAL JEC-FDTD AND JEC-CN-FDTD METHOD WITH DIFFERENT CFLNS

	JEC-FDTD	JEC-CN-FDTD			
		CFLN=1	CFLN=2	CFLN=4	CFLN=6
Time(s)	40.50	53.81	26.22	15.14	9.64
Memory(M)	82.564	93.288	93.120	92.580	92.376

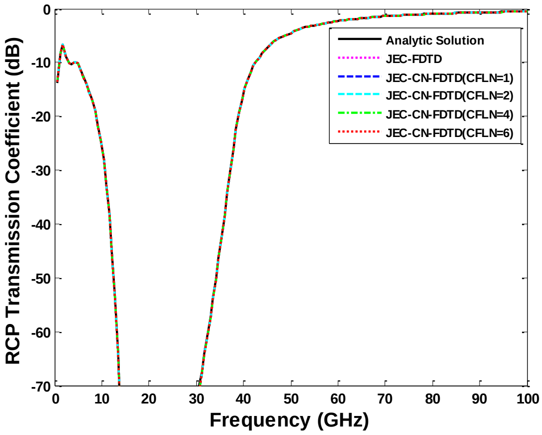

Figure 3. RCP transmission coefficients versus frequency obtained from the proposed JEC-CN-FDTD formulations with different CFLNs, the conventional JEC-FDTD and analytical solution for anisotropic magnetized plasma.

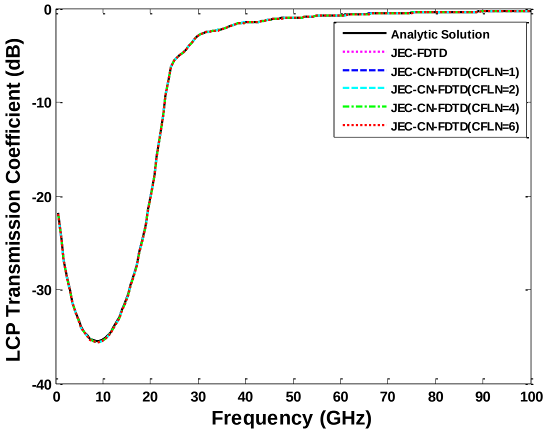

Figure 4. LCP transmission coefficients versus frequency obtained from the proposed JEC-CN-FDTD formulations with different CFLNs, the conventional JEC-FDTD and analytical solution for anisotropic magnetized plasma.

ACKNOWLEDGMENT

This work was supported by the National Natural Science Foundation of China (Grant No. 61372011).

REFERENCES

[1] F. Hunsberger, R. Luebbers, and K. Kunz, “Finite-difference time-domain analysis of gyrotropic media—I: Magnetized plasma,” *IEEE Trans. Antennas Propagat*, vol. 40, pp. 1489–1495, December 1992. “doi: 10.1109/8.204739”

[2] A. Taflove and S. C. Hagness, *Computational electrodynamics: The Finite- Difference Time-domain Method*, 3rd ed, Boston: Artechhouse, 2005. “doi: 10.1016/B978-012170960-0/50046-3”

[3] T. Namiki, “A new FDTD algorithm based on alternating-direction implicit method,” *IEEE Trans. Microw. Theory Tech*, vol. 47, pp. 2003-2007, October 1999. “doi: 10.1109/22.795075”

[4] G. Sun and C.W. Trueman, “Unconditionally stable Crank-Nicolson scheme for solving two-dimensional Maxwell's equations,” *Electron. Lett*, vol. 39, pp. 595-597, January 2003. “doi: 10.1049/el:20030416”

[5] J. Lee and B. Fornberg, “A split step approach for the 3-D Maxwell’s equations,” *J. Comput. Appl. Math.*, vol. 158, pp. 485-505, September 2003. “doi: 10.1016/S0377-0427(03)00484-9”

[6] J. Shibayama and M. Muraki, “Efficient implicit FDTD algorithm based on locally one-dimensional scheme,” *Electron. Lett*, vol. 41, pp.1046-1047, September 2005. “doi: 10.1049/el:20052381”

[7] D. J. Song, H. W. Yang, and G. B. Wang, “A research for plasma electromagnetic character using JEC-CN-FDTD algorithm based on ICCG method,” *Optik.*, vol.127, pp.1121-1125, October 2016. “doi: 10.1016/j.ijleo.2015.10.139”

[8] L. J. Xu and N. C. Yuan, “JEC-FDTD for conducting cylinder coated by anisotropic magnetized plasmas,” *IEEE Microw. Wireless Compon. Lett*, vol. 15, pp. 892–894, December 2005. “doi: 10.1109/LMWC.2005.859970”

[9] J. X. Li, Y. Yu, and X. M. Zhao, “Z-transform for unconditional stable Crank-Nicolson FDTD implementation of SC-PML for dispersive Debye media,” *Electron. Lett*, vol. 50, pp. 1959–1961, December 2014. “doi: 10.1049/el.2014.2429”

Jianxiong Li was born in Tianjin, China in 1969. He received the B.Sc. and M.Sc. degrees in physics in 1991 and 1994, respectively, and the Ph.D. degree in communication and information system in 2007, from Tianjin University, Tianjin, China. His main research interests are in computational electromagnetics, and microwave technology.

He is a Professor in School of Electronics and Information Engineering, Tianjin Polytechnic University, Tianjin, 300387, China, and also in Tianjin Key Laboratory of Optoelectronic Detection Technology and Systems, Tianjin, 300387, China

Prof. Li is senior member of China Institute of Communications.

Yongjia Zhuang was born in Mudanjiang, China in 1992. He received the B.Sc. degree in the school of Electronics and Information Engineering, Tianjin Polytechnic University, Tianjin, 300387, China.

He is currently pursuing the M.Sc. degree in the school of Electronics and Information Engineering, Tianjin Polytechnic University, Tianjin, 300387, China. His currently research interest in computational electromagnetics.